Problem 3.36

Suppose

$$\Psi(x,0) = \frac{A}{x^2 + a^2}, \quad (-\infty < x < \infty)$$

for constants A and a.

(a) Determine A, by normalizing $\Psi(x,0)$.

(b) Find $\langle x \rangle$, $\langle x^2 \rangle$, and σ_x (at time $t = 0$).

(c) Find the momentum space wave function $\Phi(p,0)$, and check that it is normalized.

(d) Use $\Phi(p,0)$ to calculate $\langle p \rangle$, $\langle p^2 \rangle$, and σ_p (at time $t = 0$).

(e) Check the Heisenberg uncertainty principle for this state.

Solution

Start by normalizing $\Psi(x,0)$.

$$1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \, dx = \int_{-\infty}^{\infty} \frac{A^2}{(x^2+a^2)^2} \, dx = A^2 \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$$

Make the following substitution.

$$x = a \tan\theta \quad \rightarrow \quad x^2 + a^2 = a^2(1 + \tan^2\theta) = a^2 \sec^2\theta$$
$$dx = a \sec^2\theta \, d\theta$$

Consequently,

$$\begin{aligned}
1 = A^2 \int_{\tan^{-1}(-\infty)}^{\tan^{-1}(\infty)} \frac{a\sec^2\theta\, d\theta}{(a^2\sec^2\theta)^2} = \frac{A^2}{a^3}\int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sec^2\theta} &= \frac{A^2}{a^3}\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta \\
&= \frac{A^2}{a^3}\int_{-\pi/2}^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\, d\theta \\
&= \frac{A^2}{2a^3}\left(\int_{-\pi/2}^{\pi/2} d\theta + \int_{-\pi/2}^{\pi/2} \cos 2\theta\, d\theta\right) \\
&= \frac{A^2}{2a^3}\left(\pi + \underbrace{\frac{1}{2}\sin 2\theta\Big|_{-\pi/2}^{\pi/2}}_{=\,0}\right) \\
&= \frac{A^2\pi}{2a^3}.
\end{aligned}$$

Solve for A.

$$A = \pm\sqrt{\frac{2a^3}{\pi}}$$

Calculate the expectation value of x at $t = 0$.

$$\langle x \rangle = \langle \Psi \,|\, \hat{x} \,|\, \Psi \rangle = \int_{-\infty}^{\infty} \Psi^*(x,0) x \Psi(x,0)\, dx = \int_{-\infty}^{\infty} \left(\frac{\sqrt{\frac{2a^3}{\pi}}}{x^2 + a^2} \right) x \left(\frac{\sqrt{\frac{2a^3}{\pi}}}{x^2 + a^2} \right) dx$$

$$= \frac{2a^3}{\pi} \underbrace{\int_{-\infty}^{\infty} \frac{x}{(x^2 + a^2)^2}\, dx}_{=0}$$

Because the integrand is an odd function and the integration is over a symmetric interval, the integral is zero.

$$\langle x \rangle = 0$$

Calculate the expectation value of x^2 at $t = 0$.

$$\langle x^2 \rangle = \langle \Psi \,|\, \hat{x}^2 \,|\, \Psi \rangle = \int_{-\infty}^{\infty} \Psi^*(x,0) x^2 \Psi(x,0)\, dx = \int_{-\infty}^{\infty} \left(\frac{\sqrt{\frac{2a^3}{\pi}}}{x^2 + a^2} \right) x^2 \left(\frac{\sqrt{\frac{2a^3}{\pi}}}{x^2 + a^2} \right) dx$$

$$= \frac{2a^3}{\pi} \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)^2}\, dx$$

Make the following substitution.

$$x = a \tan\theta \quad \rightarrow \quad x^2 + a^2 = a^2(1 + \tan^2\theta) = a^2 \sec^2\theta$$

$$dx = a \sec^2\theta\, d\theta$$

Consequently,

$$\begin{aligned}
\langle x^2 \rangle &= \frac{2a^3}{\pi} \int_{\tan^{-1}(-\infty)}^{\tan^{-1}(\infty)} \frac{(a\tan\theta)^2}{(a^2\sec^2\theta)^2} (a \sec^2\theta\, d\theta) \\
&= \frac{2a^3}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\tan^2\theta}{a\sec^2\theta}\, d\theta \\
&= \frac{2a^2}{\pi} \int_{-\pi/2}^{\pi/2} \sin^2\theta\, d\theta \\
&= \frac{2a^2}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{2}(1 - \cos 2\theta)\, d\theta \\
&= \frac{a^2}{\pi} \left(\int_{-\pi/2}^{\pi/2} d\theta - \int_{-\pi/2}^{\pi/2} \cos 2\theta\, d\theta \right) \\
&= \frac{a^2}{\pi} \Bigg(\pi - \underbrace{\frac{1}{2} \sin 2\theta \Big|_{-\pi/2}^{\pi/2}}_{=0} \Bigg) \\
&= a^2.
\end{aligned}$$

Therefore, the standard deviation in position at $t = 0$ is

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{a^2 - (0)^2} = a.$$

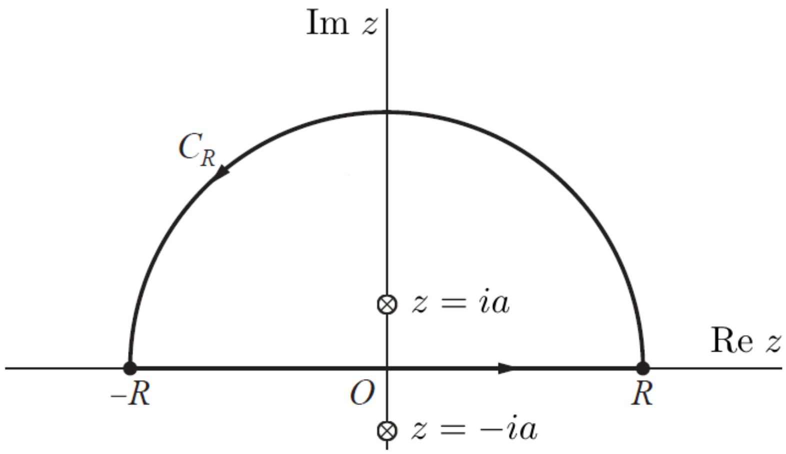
Now determine the momentum-space wave function $\Phi(p, 0)$ by taking the Fourier transform of the position-space wave function $\Psi(x, 0)$.

$$\begin{aligned}
\Phi(p,0) &= \mathscr{F}\{\Psi(x,0)\} \\
&= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\Psi(x,0)\,dx \\
&= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\left(\frac{\sqrt{\frac{2a^3}{\pi}}}{x^2+a^2}\right)dx \\
&= \frac{1}{\pi}\sqrt{\frac{a^3}{\hbar}}\int_{-\infty}^{\infty}\frac{e^{-ipx/\hbar}}{x^2+a^2}\,dx \\
&= \frac{1}{\pi}\sqrt{\frac{a^3}{\hbar}}\int_{-\infty}^{\infty}\frac{\cos\frac{px}{\hbar} - i\sin\frac{px}{\hbar}}{x^2+a^2}\,dx \\
&= \frac{1}{\pi}\sqrt{\frac{a^3}{\hbar}}\left(\int_{-\infty}^{\infty}\frac{\cos\frac{px}{\hbar}}{x^2+a^2}\,dx - i\underbrace{\int_{-\infty}^{\infty}\frac{\sin\frac{px}{\hbar}}{x^2+a^2}\,dx}_{=0}\right) \\
&= \frac{1}{\pi}\sqrt{\frac{a^3}{\hbar}}\int_{-\infty}^{\infty}\frac{\cos\left|\frac{px}{\hbar}\right|}{x^2+a^2}\,dx \\
&= \frac{1}{\pi}\sqrt{\frac{a^3}{\hbar}}\int_{-\infty}^{\infty}\frac{\cos\frac{|p||x|}{\hbar}}{x^2+a^2}\,dx \\
&= \frac{1}{\pi}\sqrt{\frac{a^3}{\hbar}}\left(\int_{-\infty}^{0}\frac{\cos\frac{-|p|x}{\hbar}}{x^2+a^2}\,dx + \int_{0}^{\infty}\frac{\cos\frac{|p|x}{\hbar}}{x^2+a^2}\,dx\right) \\
&= \frac{1}{\pi}\sqrt{\frac{a^3}{\hbar}}\left(\int_{-\infty}^{0}\frac{\cos\frac{|p|x}{\hbar}}{x^2+a^2}\,dx + \int_{0}^{\infty}\frac{\cos\frac{|p|x}{\hbar}}{x^2+a^2}\,dx\right) \\
&= \frac{1}{\pi}\sqrt{\frac{a^3}{\hbar}}\int_{-\infty}^{\infty}\frac{\cos\frac{|p|x}{\hbar}}{x^2+a^2}\,dx \\
&= \frac{1}{\pi}\sqrt{\frac{a^3}{\hbar}}\int_{-\infty}^{\infty}\frac{\operatorname{Re} e^{i|p|x/\hbar}}{x^2+a^2}\,dx \\
&= \frac{1}{\pi}\sqrt{\frac{a^3}{\hbar}}\operatorname{Re}\int_{-\infty}^{\infty}\frac{e^{i|p|x/\hbar}}{x^2+a^2}\,dx \qquad (1)
\end{aligned}$$

In order to evaluate this integral, consider

$$\int_C \frac{e^{i|p|z/\hbar}}{z^2+a^2}\,dz,$$

over the closed semicircular contour C in the complex plane shown below.



The marked points, $z = ia$ and $z = -ia$, represent the singularities of the integrand (where it blows up). Only the one at $z = ia$ is enclosed, so by Cauchy's residue theorem,

$$\oint_C \frac{e^{i|p|z/\hbar}}{z^2+a^2}\,dz = 2\pi i \operatorname*{Res}_{z=ia} \frac{e^{i|p|z/\hbar}}{z^2+a^2}.$$

C consists of the straight line segment L on the real axis and the semicircular arc C_R.

$$\begin{aligned}
\int_L \frac{e^{i|p|z/\hbar}}{z^2+a^2}\,dz + \int_{C_R} \frac{e^{i|p|z/\hbar}}{z^2+a^2}\,dz &= 2\pi i \operatorname*{Res}_{z=ia} \frac{e^{i|p|z/\hbar}}{z^2+a^2} \\
&= 2\pi i \operatorname*{Res}_{z=ia} \frac{e^{i|p|z/\hbar}}{(z+ia)(z-ia)} \\
&= 2\pi i \operatorname*{Res}_{z=ia} \frac{\phi(z)}{z-ia}, \quad \text{where } \phi(z) = \frac{e^{i|p|z/\hbar}}{z+ia}
\end{aligned}$$

$z = ia$ is a pole of order 1 (a simple pole), so the residue is $\phi(ia)$.

$$\begin{aligned}
\int_L \frac{e^{i|p|z/\hbar}}{z^2+a^2}\,dz + \int_{C_R} \frac{e^{i|p|z/\hbar}}{z^2+a^2}\,dz &= 2\pi i[\phi(ia)] \\
&= 2\pi i \left[\frac{e^{i|p|(ia)/\hbar}}{(ia)+ia}\right] \\
&= 2\pi i \left(\frac{e^{-|p|a/\hbar}}{2ia}\right) \\
&= \frac{\pi}{a} e^{-|p|a/\hbar} \qquad (2)
\end{aligned}$$

Write the parameterizations for L and C_R.

$$L: \quad z = x, \qquad -R \leq x \leq R$$

$$C_R: \quad z = Re^{i\theta}, \qquad 0 \leq \theta \leq \pi$$

As a result, equation (2) becomes

$$\int_{-R}^{R} \frac{e^{i|p|x/\hbar}}{x^2+a^2}\,(dx) + \int_0^{\pi} \frac{e^{i|p|Re^{i\theta}/\hbar}}{(Re^{i\theta})^2+a^2}(iRe^{i\theta}\,d\theta) = \frac{\pi}{a}e^{-|p|a/\hbar}$$

$$\int_{-R}^{R} \frac{e^{i|p|x/\hbar}}{x^2+a^2}\,dx + \int_0^{\pi} \frac{e^{i|p|R(\cos\theta+i\sin\theta)/\hbar}}{R^2e^{2i\theta}+a^2}(iRe^{i\theta})\,d\theta = \frac{\pi}{a}e^{-|p|a/\hbar}$$

$$\int_{-R}^{R} \frac{e^{i|p|x/\hbar}}{x^2+a^2}\,dx + \int_0^{\pi} \frac{e^{i|p|R\cos\theta/\hbar}e^{-|p|R\sin\theta/\hbar}}{R^2e^{2i\theta}+a^2}(iRe^{i\theta})\,d\theta = \frac{\pi}{a}e^{-|p|a/\hbar}. \tag{3}$$

Consider the magnitude of the second integral. Note that for any two complex numbers, $|z_1 + z_2| \geq ||z_1| - |z_2||$.

$$\begin{aligned}
\left|\int_0^{\pi} \frac{e^{i|p|R\cos\theta/\hbar}e^{-|p|R\sin\theta/\hbar}}{R^2e^{2i\theta}+a^2}(iRe^{i\theta})\,d\theta\right| &\leq \int_0^{\pi} \left|\frac{e^{i|p|R\cos\theta/\hbar}e^{-|p|R\sin\theta/\hbar}}{R^2e^{2i\theta}+a^2}(iRe^{i\theta})\right| d\theta \\
&= \int_0^{\pi} \frac{e^{-|p|R\sin\theta/\hbar}}{|R^2e^{2i\theta}+a^2|}(R)\,d\theta \\
&\leq \int_0^{\pi} \frac{1}{||R^2e^{2i\theta}| - |a^2||}(R)\,d\theta \\
&= \int_0^{\pi} \frac{1}{R^2-a^2}(R)\,d\theta \\
&= \frac{R}{R^2-a^2}\int_0^{\pi} d\theta \\
&= \frac{\pi R}{R^2-a^2}
\end{aligned}$$

In the limit as $R \to \infty$, this upper bound of the magnitude tends to zero. Therefore, taking the limit as $R \to \infty$, equation (3) becomes

$$\int_{-\infty}^{\infty} \frac{e^{i|p|x/\hbar}}{x^2+a^2}\,dx = \frac{\pi}{a}e^{-|p|a/\hbar},$$

and the momentum-space wave function in equation (1) becomes

$$\begin{aligned}
\Phi(p,0) &= \frac{1}{\pi}\sqrt{\frac{a^3}{\hbar}}\,\mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{i|p|x/\hbar}}{x^2+a^2}\,dx \\
&= \frac{1}{\pi}\sqrt{\frac{a^3}{\hbar}}\left(\frac{\pi}{a}e^{-|p|a/\hbar}\right) \\
&= \sqrt{\frac{a}{\hbar}}e^{-|p|a/\hbar}.
\end{aligned}$$

Check that it's normalized.

$$\begin{aligned}
\int_{-\infty}^{\infty} |\Phi(p,0)|^2\,dp &= \int_{-\infty}^{\infty} \Phi^*(p,0)\Phi(p,0)\,dp \\
&= \int_{-\infty}^{\infty} \left(\sqrt{\frac{a}{\hbar}}e^{-|p|a/\hbar}\right)\left(\sqrt{\frac{a}{\hbar}}e^{-|p|a/\hbar}\right) dp \\
&= \frac{a}{\hbar}\int_{-\infty}^{\infty} e^{-2|p|a/\hbar}\,dp \\
&= \frac{a}{\hbar}\left(\int_{-\infty}^{0} e^{2pa/\hbar}\,dp + \int_{0}^{\infty} e^{-2pa/\hbar}\,dp\right) \\
&= \frac{a}{\hbar}\left[\frac{\hbar}{2a}e^{2pa/\hbar}\bigg|_{-\infty}^{0} + \left(-\frac{\hbar}{2a}\right)e^{-2pa/\hbar}\bigg|_{0}^{\infty}\right] \\
&= \frac{a}{\hbar}\left[\frac{\hbar}{2a}(e^0 - e^{-\infty}) - \frac{\hbar}{2a}(e^{-\infty} - e^0)\right] \\
&= 1
\end{aligned}$$

Calculate the expectation value of p at $t = 0$.

$$\begin{aligned}
\langle p\rangle = \langle \Phi \,|\, \hat{p} \,|\, \Phi\rangle = \int_{-\infty}^{\infty} \Phi^*(p,0)p\Phi(p,0)\,dx &= \int_{-\infty}^{\infty} \left(\sqrt{\frac{a}{\hbar}}e^{-|p|a/\hbar}\right) p \left(\sqrt{\frac{a}{\hbar}}e^{-|p|a/\hbar}\right) dp \\
&= \frac{a}{\hbar}\underbrace{\int_{-\infty}^{\infty} pe^{-2|p|a/\hbar}\,dp}_{=\,0}
\end{aligned}$$

Because the integrand is an odd function and the integration is over a symmetric interval, the integral is zero.

$$\langle p\rangle = 0$$

Calculate the expectation value of p^2 at $t = 0$.

$$\begin{aligned}
\langle p^2\rangle = \langle \Phi \,|\, \hat{p}^2 \,|\, \Phi\rangle = \int_{-\infty}^{\infty} \Phi^*(p,0)p^2\Phi(p,0)\,dx &= \int_{-\infty}^{\infty} \left(\sqrt{\frac{a}{\hbar}}e^{-|p|a/\hbar}\right) p^2 \left(\sqrt{\frac{a}{\hbar}}e^{-|p|a/\hbar}\right) dp \\
&= \frac{a}{\hbar}\int_{-\infty}^{\infty} p^2 e^{-2|p|a/\hbar}\,dp \\
&= \frac{a}{\hbar}\left(2\int_{0}^{\infty} p^2 e^{-2|p|a/\hbar}\,dp\right) \\
&= \frac{2a}{\hbar}\int_{0}^{\infty} p^2 e^{-2pa/\hbar}\,dp \\
&= \frac{2a}{\hbar}\int_{0}^{\infty} p^2 \frac{d}{dp}\left(-\frac{\hbar}{2a}e^{-2pa/\hbar}\right) dp
\end{aligned}$$

Continue the simplification by integrating by parts twice.

$$\begin{aligned}
\langle p^2\rangle &= \frac{2a}{\hbar}\left[\underbrace{p^2\left(-\frac{\hbar}{2a}e^{-2pa/\hbar}\right)\Big|_0^\infty}_{=0} - \int_0^\infty \frac{d}{dp}(p^2)\left(-\frac{\hbar}{2a}e^{-2pa/\hbar}\right)dp\right] \\
&= \frac{2a}{\hbar}\left[-\int_0^\infty (2p)\left(-\frac{\hbar}{2a}e^{-2pa/\hbar}\right)dp\right] \\
&= 2\int_0^\infty pe^{-2pa/\hbar}\,dp \\
&= 2\int_0^\infty p\frac{d}{dp}\left(-\frac{\hbar}{2a}e^{-2pa/\hbar}\right)dp \\
&= 2\left[\underbrace{p\left(-\frac{\hbar}{2a}e^{-2pa/\hbar}\right)\Big|_0^\infty}_{=0} - \int_0^\infty \frac{d}{dp}(p)\left(-\frac{\hbar}{2a}e^{-2pa/\hbar}\right)dp\right] \\
&= 2\left[-\int_0^\infty (1)\left(-\frac{\hbar}{2a}e^{-2pa/\hbar}\right)dp\right] \\
&= \frac{\hbar}{a}\int_0^\infty e^{-2pa/\hbar}\,dp \\
&= \frac{\hbar}{a}\left(-\frac{\hbar}{2a}e^{-2pa/\hbar}\right)\Big|_0^\infty \\
&= \frac{\hbar}{a}\left[-\frac{\hbar}{2a}(e^{-\infty}-e^0)\right] \\
&= \frac{\hbar^2}{2a^2}
\end{aligned}$$

Therefore, the standard deviation in momentum at $t = 0$ is

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{\hbar^2}{2a^2} - (0)^2} = \frac{\hbar}{\sqrt{2}a}.$$

Finally, check to see that Heisenberg's uncertainty principle is satisfied at $t = 0$.

$$\sigma_x\sigma_p = (a)\left(\frac{\hbar}{\sqrt{2}a}\right) = \frac{\hbar}{\sqrt{2}} \geq \frac{\hbar}{2}$$